International Journal of Theoretical Physics, Vol. 35, No. 10, 1996

Canonical Superenergy Tensors in General Relativity

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Received October 30, 1995

We present the concept of superenergy tensors in the framework of general relativity (GR). These tensors were introduced constructively by the author years ago and they were obtained by a suitable averaging of the energy-momentum tensors or pseudotensors. Because in GR the Einstein canonical energy-momentum pseudotensor ${}_{\mathrm{E}}t_i{}^k$ of the gravitational field and the canonical energy-momentum complex ${}_{\mathrm{E}}K_i{}^k = \sqrt{|g|}(T_i^k + {}_{\mathrm{E}}t_i{}^k)$, matter and gravitation, *are physically distinguished*, we confine this paper to the *canonical superenergy tensor* ${}^gS_i^k$ of the gravitational field Γ^i_{kl} and to the canonical total superenergy tensor $S_i^k = {}^gS_i^k + {}^mS_i^k$ of matter and gravitation only. These superenergy tensors can be obtained by the above-mentioned averaging of the pseudotensor ${}_{\mathrm{E}}t_i{}^k$ and complex ${}_{\mathrm{E}}K_i^k$. We give the analytic forms of these two canonical superenergy tensors and show some of their possible applications in GR. The canonical superenergy tensor ${}^gS_i^k$ of the gravitational field Γ^i_{kl} can be used as a *substitute* for the nonexisting energy-momentum tensor of this field.

1. INTRODUCTION

The very difficult problem of the conservation laws in general relativity (GR) has been intensively studied by many authors (see, e.g., Trautman, 1962; Cattaneo, 1965; Møller, 1966, 1978; Garecki, 1973; Goldberg, 1980; Winicour, 1980; Landau and Lifschitz, 1988; Wald, 1984; Thirring, 1978). The main results of these investigations are the following:

1. Owing to the nontensorial character of the gravitational strengths, the gravitational field in GR has no energy-momentum tensor. It follows from this that the gravitational energy and momentum are not localized. This leads us to conceptual and interpretational difficulties (Trautman, 1962; Cattaneo, 1965).

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0020-7748/96/1000-2195$09.50/0

2. Without introducing any supplementary elements into GR, one can only introduce the so-called *double-index gravitational energy-momentum pseudotensors* and also nontensorial *double-index energy-momentum complexes* (matter and gravitation).

The best of the possible gravitation double-index energy-momentum pseudotensors and complexes is the *canonical energy-momentum pseudotensor* ${}_{\mathrm{E}}t_i^{\ k}$ proposed by Einstein and connected with it the *canonical, double-index energy-momentum complex* ${}_{\mathrm{E}}K_i^{\ k} = \sqrt{|g|}(T_i^k + {}_{\mathrm{E}}t_i^{\ k})$, satisfying

$$\sqrt{|g|}(T_i^k + {}_{\mathrm{E}}t_i^{\ k}) = {}_{\mathrm{F}}U_i^{\ kl}{}_{,l} \tag{1.1}$$

and (local conservation laws)

$$[\sqrt{|g|}(T_i^k + {}_{\mathrm{E}}t_i^{\ k})]_{,k} = 0 \tag{1.2}$$

where ${}_{\mathrm{F}}U_i^{\ kl} = (-)_{\mathrm{F}}U_i^{\ lk}$ are the so-called *Freud superpotentials.*

Here T_i^k denotes the components of the symmetric energy-momentum tensor of matter and g is the determinant of the metric tensor; $,i$ denotes partial differentiation. The Latin indices run over 0, 1, 2, 3 and the symbol $:=$ means "by definition."

The equations (1.1) may be obtained by a suitable transformation of the Einstein equations with the T_i^k as sources (see, e.g., Cattaneo, 1965; Møller, 1966, 1978; Garecki, 1973; Goldberg, 1980).

The Einstein canonical energy-momentum pseudotensor ${}_{\mathrm{E}}t_i^{\ k}$ is a *quadratic function* of the gravitational strengths Γ^i_{ik} and has the following form (see, e.g., Weber, 1961; Landau and Lifschitz, 1988):

$$\begin{aligned} {}_{\mathrm{E}}t_i^{\ k} = \frac{c^4}{16\pi G}\Bigg\{ & \delta_i^k g^{ms}(\Gamma^l_{mr}\Gamma^r_{sl} - \Gamma^r_{ms}\Gamma^l_{rl}) \\ & + g^{ms}{}_{,i}\Bigg[\Gamma^k_{ms} - \frac{1}{2}(\Gamma^k_{tp}g^{tp} - \Gamma^l_{tl}g^{kt})g_{ms} \\ & - \frac{1}{2}(\delta^k_s\Gamma^l_{ml} + \delta^k_m\Gamma^l_{sl})\Bigg]\Bigg\} \end{aligned} \tag{1.3}$$

The Einstein canonical energy-momentum complex ${}_{\mathrm{E}}K_i^{\ k}$ and the pseudotensor ${}_{\mathrm{E}}t_i^{\ k}$, like other double-index energy-momentum complexes, can be reasonably used only in the case of a so-called *closed system*, i.e., in the case of an isolated and nonradiating system (Trautman, 1962; Møller, 1966, 1978), to calculate global energy and momentum (Einstein–Klein theorem) and, in general, in the case of the so-called *asymptotically flat space-times* (Goldberg, 1980; Winicour, 1980) and only by using asymptotically flat coordinates (at spatial infinity or at null infinity). By using the Einstein complex ${}_{\mathrm{E}}K_i^{\ k}$ and pseudotensor ${}_{\mathrm{E}}t_i^{\,k}$ in this way one can obtain all physically valid results obtained recently in other ways in the energy-momentum problem in GR, e.g., the ADM energy or Witten's theorem on the positivity of the total energy (Goldberg, 1980; Faddeev, 1982). On the other hand, the complex ${}_{\mathrm{E}}K_i^{\,k}$ and the pseudotensor ${}_{\mathrm{E}}t_i^{\,k}$ *have no physical meaning in a local analysis of the gravitational field.*

3. By introducing *supplementary elements* into the structure of the GR like arbitrary vector field, tetrads, a double metric, or torsion, one can obtain covariant energy-momentum complexes and covariant conservation laws.

Following this approach, the best solution of the energy-momentum (and angular momentum) problem was obtained by Komar (1959, 1962) in terms of so-called *single-index complexes* (Trautman, 1962; Cattaneo, 1965).

The single-index complexes can be best adapted to the general covariance of GR, e.g., the single-index complex obtained by Komar

$$\sqrt{|g|}T^i_k\xi^k + {}_{\mathrm{g}}J^i = \frac{c^4}{4\pi G}\,\partial_k(\sqrt{|g|}\nabla^{[i}\xi^{k]}) =: {}_{\mathrm{K}}J^i \tag{1.4}$$

satisfying *local conservation laws*

$${}_{\mathrm{K}}J^i{}_{,i} = \nabla_{i\mathrm{K}}J^i = 0 \tag{1.5}$$

is a *vector density* and therefore it leads to *geometrically correct integrals* on global quantities: the suitable integrals are scalars. Here $\xi^i(x)$ denotes an *arbitrary vector field* and ${}_{\mathrm{g}}J^i$ means the *gravitational part* of the Komar complex.

Moreover, the single-index complexes admit a formulation of the conservation laws for the angular momentum also.

However, using Komar's (or other) single-index complex ${}_{\mathrm{K}}J^i$, we have a difficult problem: how do we choose the suitable vector field $\xi^i(x)$, called the *descriptor,* in order to obtain physically valid quantities like energy and linear and angular momentum and physically valid conservation laws for these quantities?

It is assumed (Trautman, 1962; Cattaneo, 1965; Goldberg, 1980; Winicour, 1980; Wald, 1984; Komar, 1959, 1962) that, in analogy with the flat Minkowskian space-time, only the *Killing vector fields* or, at least, *asymptotically Killing vector fields* should be used as descriptors. But such vector fields do not exist in realistic space-time. Moreover, the Komar expression *needs a null-hypersurface-dependent modification* at null infinity (Goldberg, 1980; Winicour, 1980) to obtain Bondi's results, considered as correct.

To sum up, one can say that the energy-momentum problem in GR has *no satisfactory local solution.* There exists only a satisfactory global solution of the problem in asymptotically flat space-times and it *univocally distinguishes* the Einstein canonical energy-momentum complex ${}_{\mathrm{E}}K_i^k$ and the canonical pseudotensor ${}_{\mathrm{E}}t_i^k$ as the best ones.

Only the Einstein complex is derivable from the Lagrangian as the canonical one and gives correct physical results at both spatial and null infinity (Goldberg, 1980).

Therefore, in the following we will confine ourselves to the canonical pseudotensor ${}_{\mathrm{E}}t_i^k$ and to the canonical complex ${}_{\mathrm{E}}K_i^k$ only.

2. THE CANONICAL SUPERENERGY TENSORS IN GENERAL RELATIVITY

The difficulties connected with localization of the gravitational energy and especially the lack of any gravitational energy-momentum tensor in GR inclined me years ago (e.g., Garecki, 1981a) to introduce *superenergy tensors* into GR.

The idea of the superenergy tensors uses some special properties of the so-called *normal coordinates* NCS(P) (the point P is the origin of these coordinates) in the framework of Riemann geometry (Veblen, 1933; Schouten, 1954; Petrov, 1966). The most important of these properties discovered by Veblen (1933) and Schouten (1954) is the following: the coefficients of the Maclaurin series formed in NCS(P) for tensorial fields or pseudotensorial fields formed from gravitational strengths Γ^i_{kl} *are true tensors*. Moreover, the construction of the superenergy tensors uses some generalization of the Pirani averaging (Pirani, 1957).

Let us suppose that in the space-time V_4 a physical field Φ is given. Its invariant Lagrangian density is Λ and the symmetric, metric energy-momentum tensor corresponding to it is ${}^{\Phi}T_i^k$. We will construct the *metric superenergy tensor* ${}^{\Phi}S_i^k(P)$ of the field Φ and at the point $P \in V_4$ in the following manner.

Let us introduce the normal coordinate system (Veblen, 1933; Schouten, 1954; Petrov, 1966) NCS(P) for the Riemannian connection Γ^i_{kl} and consider the four-dimensional cube C: $(-)a \leq y^i \leq a$ with $a > 0$ being sufficiently small. Here y^i $(i = 0, 1, 2, 3)$ denotes the normal coordinates. The point P [= the origin of the NCS(P)] is the geometric center of this cube.

Then we define

$$ {}^{\Phi}S_i^k(P) := \lim_{a\to 0} \frac{\int_{(-)a}^{a}\int_{(-)a}^{a}\int_{(-)a}^{a}\int_{(-)a}^{a} ({}^{\Phi}T_i^k - {}^{\Phi}\mathring{T}_i^k)\, dy^0 dy^1 dy^2 dy^3}{8/3a^6} \tag{2.1} $$

If the field ${}^{\Phi}T_i^k$ is of the class[2] C^r, $r \geq 3$, then we have, by expanding ${}^{\Phi}T_i^k$ to the third order according to Maclaurin's formula,

[2] In the following we will assume that all the considered fields are of the class C^r, $r \geq 3$.

$$^{\Phi}T_i^k = {}^{\Phi}\dot{T}_i^k + \partial_l\, {}^{\Phi}\dot{T}_i^k y^l + \frac{1}{2}\,\partial_a\partial_b\, {}^{\Phi}\dot{T}_i^k y^a y^b + R_3 \tag{2.2}$$

Here R_3 is the remainder of the third order.

Substituting this into (2.1), we get after some calculations

$$^{\Phi}S_i^k(P) = \partial_0\partial_0\, {}^{\Phi}\dot{T}_i^k + \partial_1\partial_1\, {}^{\Phi}\dot{T}_i^k + \partial_2\partial_2\, {}^{\Phi}\dot{T}_i^k + \partial_3\partial_3\, {}^{\Phi}\dot{T}_i^k \tag{2.3}$$

Introducing

$$\dot{u}^l = \delta_0^l \quad \text{in NCS(P)}, \qquad \dot{g}^{ab} = \eta^{ab} \quad \text{in NCS(P)} \tag{2.4}$$

where η^{ab} is the Minkowskian metric with signature $(+, -, -, -)$, we can write this covariantly as

$$\begin{aligned} ^{\Phi}S_i^k(P;\, u^l) &= (2\dot{u}^a\dot{u}^b - \dot{g}^{ab})\,{}^{\Phi}\dot{T}_i^k{}_{,ab} \\ &=: (2\dot{u}^a\dot{u}^b - \dot{g}^{ab})\,{}^{\Phi}\dot{T}_i^k{}_{ab} \end{aligned} \tag{2.5}$$

The dot above a tensor field denotes the value of this field at the point P and the symbols $,i$ or ∂_i denote partial differentiation.

The ${}^{\Phi}\dot{T}_i^k{}_{ab}$ is a true tensor (Veblen, 1933; Schouten, 1964; Petrov, 1966) having the following form (Garecki, 1981a):

$$^{\Phi}\dot{T}_i^k{}_{ab} = \nabla_{(a}\nabla_{b)}^{\Phi}\dot{T}_i^k + \frac{1}{3}\,\dot{R}^c{}_{(a|i|b)}\dot{T}_c^k - \frac{1}{3}\,\dot{R}^k{}_{(a|c|b)}\dot{T}_i^c \tag{2.6}$$

Here ∇ denotes the covariant derivative with respect to the Riemann connection $\omega^k{}_k = \Gamma^i_{kl}\, dx^l$ and parentheses mean symmetrization, $(a|c|b) := \frac{1}{2}(acb + bca)$.

We will call the four-index tensor given by (2.6) the *metric superenergy supertensor* of the field Φ. This tensor is more fundamental than the double-index tensor ${}^{\Phi}S_i^k(P;\, u^l)$.

The averaging given by (2.1) is a generalization of the averaging proposed by Pirani (1957).

The superenergy tensor ${}^{\Phi}S_i^k(P;\, u^l)$ is a local construction which explicitly depends on the form of the energy-momentum tensor ${}^{\Phi}T_i^k$ and on the four-velocity u^l: $u^l u_l = 1$ of an observer at rest at the origin of the NCS(P).

For a given energy-momentum tensor ${}^{\Phi}T_i^k$ the field of the supertensor ${}^{\Phi}T_i^k{}_{ab}$ is uniquely determined. The same can be done for the tensor field ${}^{\Phi}S_i^k(P;\, u^l)$ (P is variable now) provided a vector field u^l is given.

Further, we define the density ϵ_s of the superenergy of the field Φ and Poynting's supervector P_i of this field for an observer at rest at the origin of the NCS(P),

$$\epsilon_s := {}^{\Phi}\dot{S}_i^k\dot{u}^i\dot{u}_k \tag{2.7}$$

$$P_i := (\delta_i^k - \dot{u}^k\dot{u}_i)\, {}^{\Phi}\dot{S}_k{}^l\dot{u}_l \tag{2.8}$$

The components of the superenergy tensor ${}^{\Phi}S_i^{\,k}(P;\, u^l)$ have dimensions equal to $\mathrm{cm}^{(-)2}\times$ dimensions of the components of the energy-momentum tensor. The same dimensions also apply to the components of the superenergy supertensor ${}^{\Phi}T_i^{\,k}{}_{ab}$.

If we substitute into (2.1), after expanding according to Maclaurin's formula, the so-called *energy-momentum pseudotensors of the gravitational field* of GR, then we will also get true tensors. These tensors may have more symmetry properties than the pseudotensors from which they were obtained, e.g., *the canonical superenergy tensor* ${}^{g}S_i^{\,k}(P;\, u^l)$ of the gravitational field Γ^i_{kl} obtained from the canonical pseudotensor ${}_{\mathrm{E}}t_i^{\,k}$ is (in vacuum) symmetric (after raising or lowering a suitable index).

This canonical superenergy tensor of the gravitational field has the following form (Garecki, 1981a):

$$\begin{aligned}
{}^{g}\dot{S}_i^{\,k}(P;\, u^l) &= \frac{2\alpha}{9}\,(2\dot{u}^a\dot{u}^b - \dot{g}^{ab})[\dot{B}^k{}_{iab} + \dot{T}^k{}_{iab} \\
&\quad - \frac{1}{2}\,\delta_i^k\dot{R}^{lmn}{}_b(\dot{R}_{lmna} + \dot{R}_{lnma}) \\
&\quad + 2\delta_i^k\beta^2\dot{E}_{(a|g}\dot{E}^g{}_{|b)} - 3\beta^2\dot{E}_{i(a|}\dot{E}^k{}_{|b)} \\
&\quad - \beta\dot{R}_{gi}{}^k{}_{(a}\dot{E}^g{}_{b)} - \beta\dot{R}^k{}_{ig(a}\dot{E}^g{}_{b)}] \\
&=: (2\dot{u}^a\dot{u}^b - \dot{g}^{ab})G_i^{\,k}{}_{ab}
\end{aligned} \tag{2.9}$$

In the above expression $\alpha = c^4/(16\pi G) = 1/(2\beta)$; R^i_{klm} denotes the curvature tensor components and E_i^k are the components of the modified energy-momentum tensor of matter,

$$E_i^k := T_i^k - \frac{1}{2}\,\delta_i^k T_l^l = T_i^k - \frac{1}{2}\,\delta_i^k T$$

Here T_i^k denotes the components of a symmetric, metric energy-momentum tensor of matter as sources in the Einstein equations.

The tensor $G_i^{\,k}{}_{ab}$ is the *canonical superenergy supertensor* of the gravitational field Γ^a_{bc}. Here $B^k{}_{iab}$ denotes the *Bel–Robinson tensor* components (Bel, 1958, 1959, 1962)

$$B^k{}_{iab} := R^{klm}{}_a R_{ilmb} + R^{klm}{}_b R_{ilma} - \frac{1}{2}\,\delta_i^k R^{lmn}{}_a R_{lmnb} \tag{2.10}$$

and

$$T^k{}_{iab} := R^{klm}{}_a R_{imlb} + R^{klm}{}_b R_{imla} - \frac{1}{2}\,\delta_i^k R^{lmn}{}_a R_{lnmb} \tag{2.11}$$

In vacuum the superenergy tensor ${}^{g}S_i^{\,k}(P;\, u^l)$ has the simpler form

$$^{g}S_i^{\ k}(P;\ u^l) = \frac{2\alpha}{9}\,(2\dot{u}^a\dot{u}^b - \dot{g}^{ab})[\dot{B}^k{}_{iab} + \dot{T}^k{}_{iab}] \tag{2.12}$$

$B^k{}_{iab}$ denotes the components of the vacuum Bel–Robinson tensor

$$B^k{}_{iab} := W^{klm}{}_a W_{ilmb} + W^{klm}{}_b W_{ilma} - \frac{1}{2}\,\delta_i^k W^{lmn}{}_a W_{lmnb} \tag{2.13}$$

$T^k{}_{iab}$ is the tensor

$$T^k{}_{iab} := W^{klm}{}_a W_{imlb} + W^{klm}{}_b W_{imla} - \frac{1}{4}\,\delta_i^k g_{ab} W^{lmnp} W_{lmnp} \tag{2.14}$$

and $W^k{}_{iab}$ denotes the components of the Weyl tensor.

The superenergy tensors of the gravitational field Γ^i_{kl} calculated from other energy-momentum pseudotensors of the field are much more complicated in form then the canonical one given by (2.9) (Garecki, 1973). However, every superenergy tensor of the gravitational field Γ^i_{kl} *explicitly contains* the tensor (2.9) (Garecki, 1973). This points out the special physical meaning of the canonical superenergy tensor $^{g}S_i^{\ k}(P;\ u^l)$ of the field.

It is easily seen that the gravitational superenergy density $^{g}\epsilon_s := {}^{g}S_i^{\ k}u^i u_k$ *is localized.*

We may call the tensor

$$M_i^{\ k}(P;\ u^l) := \frac{2\alpha}{9}\,(2\dot{u}^a\dot{u}^b - \dot{g}^{ab})\dot{B}^k{}_{iab} \tag{2.15}$$

the Maxwellian part of the canonical superenergy tensor $^{g}S_i^{\ k}(P;\ u^l)$ of the gravitational field Γ^i_{kl} because the method of construction of the Bel–Robinson tensor $B^k{}_{iab}$ from the Bianchi identities and their contractions (see, e.g., Öktem, 1968; Garecki, 1973) is identical to the method of obtaining the symmetry energy-momentum tensor of the electromagnetic field by using the Maxwell equations only.

We remark here that for the vacuum metric of types II and III in the Petrov algebraic classification (Petrov, 1966)

$$^{g}S_0^{\ k} = \frac{c^4}{216\pi G}\,B^k{}_{000} \tag{2.16}$$

According to *Pirani's criterion* (Pirani, 1957), gravitational radiation is present in the empty space-times of types II and III.

Thus, the formula (2.16) shows that the *local superenergy flux* of a gravitational wave *is proportional* to the components $B^k{}_{000}$ of the Bel–Robinson tensor.

In analogy to the canonical energy-momentum complex $_{E}K_i^{\ k} = \sqrt{|g|}(T_i^{\ k} + {}_{E}t_i^{\ k})$ of matter and gravitation, one can consider in GR the *total,* *canonical superenergy tensor* $S_i^k(P;\, u^l)$ of matter and gravitation (see, e.g., Garecki, 1981a). This tensor should be calculated from the sum $T_i^k + {}_E t_i^k$ by averaging according to the formula (2.1). After some calculations we get the obvious result

$$S_i^k(P;\, u^l) = {}^{g}S_i^k(P;\, u^l) + {}^{m}S_i^k(P;\, u^l) \tag{2.17}$$

where the canonical superenergy tensor ${}^{g}S_i^k(P;\, u^l)$ of the gravitational field is given by the formula (2.9) and the superenergy tensor of matter ${}^{m}S_i^k(P;\, u^l)$ has the following form:

$$\begin{aligned} {}^{m}S_i^k(P;\, u^l) := (2\dot{u}^a\dot{u}^b - \dot{g}^{ab})\Big[\nabla_{(a}\nabla_{b)}\dot{T}_i^k + \frac{1}{3}\dot{R}^c{}_{(a|i|b)}\dot{T}_c^k \\ - \frac{1}{3}\dot{R}^k{}_{(a|c|b)}\dot{T}_i^c\Big] \end{aligned} \tag{2.18}$$

The total canonical superenergy tensor, matter and gravitation, $S_i^k(P;\, u^l)$ allows us to introduce the notion of *the global superenergetic quantities* S_i *of a closed system*. We define these quantities in analogy to the *energetic* integrals, in the following manner:

$$S_i(\Sigma) := \int_\Sigma S_i^k \sqrt{|g|}\, d\sigma_k \tag{2.19}$$

where S_i^k denotes the total canonical superenergy tensor of matter and gravitation given by (2.17), (2.9), and (2.18) and Σ is a spacelike hypersurface which is asymptotically flat. $d\sigma_i$ is an integration element (see, e.g., Landau and Lifschitz, 1988).

We fix the vector field u^l: $u^l u_l = 1$ appearing in $S_i^k(P;\, u^l)$ in the following way: we put the unit timelike basic vector of every NCS(P) (P is variable now) proportional to the timelike vector of the natural frame at the point P of the used global coordinates. Physically this means that we use as the vector field u^l the field of the four-velocities of the observers at rest with respect to the used global coordinates. This is a natural choice of the field u^l in fixed, global coordinates if we want to have a uniquely determined tensor field $S_i^k(x)$ to calculate the global superenergetic quantities of a closed system.

In asymptotically Lorentzian coordinates (ct, x, y, z) and if Σ is $x^0 = ct = \text{const}$, the integrals (2.19) take the form

$$S_i = \int_{x^0=\text{const}} S_i^0 \sqrt{|g|}\, dx\, dy\, dz \tag{2.20}$$

The integrals $S_i(\Sigma)$, for a fixed Σ, form a free vector with respect to $GL(4;\, R)$; moreover, the integral

$$S_0(V) := \int_V \epsilon_s \sqrt{|g|}\, dx\, dy\, dz = \int_V ({}^g\epsilon_s + {}^m\epsilon_s)\sqrt{|g|}\, dx\, dy\, dz$$

is invariant with respect to arbitrary coordinate transformations. From the last remark it follows that the *amount of the superenergy inside the volume* V has a physical meaning.

3. APPLICATIONS OF THE CANONICAL SUPERENERGY TENSORS IN GENERAL RELATIVITY

The canonical superenergy tensor ${}^gS_i^{\,k}(P; u^l)$ of the gravitational field Γ^i_{kl} is very useful for local analysis of the vacuum solutions to the Einstein equations, which are interpreted as representing *gravitational waves* (see, e.g., Garecki, 1980, 1981a,b; Krawczak, 1982; Kościelak, 1987). Analysis of the so-called *plane-fronted waves* and *plane waves* showed that these waves transfer a superenergy flux and have positive-definite superenergy densities. On the other hand, the plane-fronted waves and plane waves *cannot transfer any energy flux* because the canonical pseudotensor ${}_Et_i^{\,k}$ *vanishes identically* for such solutions to the vacuum Einstein equations (Dańczura, 1984).

In Garecki (1981a) we examined the integral superenergetic quantities for a closed system, and calculated the appropriate quantities for the Schwarzschild space-time.

It is very interesting that the gravitational superenergy density of the exterior Schwarzschild space-time is *positive-definite* and has the very simple form

$$^g\epsilon_s = \frac{8}{3}\,\alpha\,\frac{r_g^2}{r^6} > 0 \tag{3.1}$$

where $r_g := 2GM/c^2$ and $r > r_*$. Here M denotes the total mass of the spherical star and $r = r_*$ is the value of the radial coordinate r corresponding to the surface of the star.

From (3.1) we obtain that the total gravitational superenergy ${}^gS$ of the exterior Schwarzschild space-time defined as

$$^gS := \int_{r=r_*}^{\infty} \int_0^{\pi} \int_0^{2\pi} {}^g\epsilon_s \sqrt{|g^{\text{ext}}|}\, dr\, d\theta\, d\varphi \tag{3.2}$$

is equal to

$$^gS = \frac{32\pi}{9}\,\alpha\,\frac{r_g^2}{r_\star^3} > 0 \tag{3.3}$$

In general, the integrals representing global superenergetic quantities of a closed system *are convergent* and if space-time is *static* or *stationary*, then they are independent of time (Garecki, 1981a).

In Garecki (1981a) the physical meaning and the Newtonian limit of the gravitational superenergy were also considered. It was shown that in the Newtonian limit the canonical gravitational superenergy tensor is an energy-momentum tensor for the field of tidal forces.

In Garecki (1993) we calculated the gravitational and total superenergy densities and the total superenergy[3] of a closed Friedmann radiation-dominated universe (FRDU). The restriction to a closed FRDU is not essential. It was only done because in that case there exists a *direct*, i.e., *nonparametric* solution $R = R(t)$ to the Friedmann equations. R denotes the so-called *scale factor* and the parameter t is the *cosmic time* (see, e.g., Garecki, 1993).

It has been shown in this paper that the superenergy densities ${}^{g}\epsilon_s$ and ${}^{m}\epsilon_s$ are *positive-definite* for $R > 0$ and have singularities if $R \to 0^+$. Also *positive-definite* and *finite* for $R > 0$ is the total superenergy S, matter and gravitation, of the FRDU, defined by the integral

$$S = \int_{t=\text{const}} ({}^{g}\epsilon_s + {}^{m}\epsilon_s)\sqrt{|g|}\, d\chi\, d\vartheta\, d\varphi \tag{3.4}$$

If we do analogous calculations for *open Friedmann models* having the *curvature index* $k = 0$,[4] then we obtain the following very simple results (from now on we put $c = 1$):

1. *Friedmann matter-dominated universe* (FMDU) ($k = 0$, $p = 0$, where p denotes the rest pressure of matter):

$${}^{g}\epsilon_s = \frac{11}{162\pi G t^4} > 0 \tag{3.5}$$

$${}^{m}\epsilon_s = \frac{55}{27\pi G t^4} > 0, \tag{3.6}$$

$$\epsilon_s := {}^{g}\epsilon_s + {}^{m}\epsilon_s = \frac{341}{162\pi G t^4} > 0 \tag{3.7}$$

2. *Friedmann radiation dominated universe* (FRDU) ($k = 0$, $p = \epsilon/3$, where p is the rest pressure and ϵ is the rest energy density of matter):

[3] As the field u^i we have taken the four-velocity field of the so-called *isotropic* or *fundamental observers* at rest in the used comoving coordinates. This field is *geometrically and physically* distinguished in this case.

[4] Such cosmological models give the best mathematical models of the universe having an *inflation phase*.

$$ {}^{g}\epsilon_s = \frac{1}{96\pi G t^4} > 0 \tag{3.8} $$

$$ {}^{m}\epsilon_s = \frac{63}{64\pi G t^4} > 0 \tag{3.9} $$

$$ \epsilon_s := {}^{g}\epsilon_s + {}^{m}\epsilon_s = \frac{191}{192\pi G t^4} > 0 \tag{3.10} $$

We see that the all canonical superenergy densities are *positive definite* and *tend to zero* if the cosmic time t goes to infinity. If the cosmic time $t \to 0^+$ (this is equivalent to $R \to 0^+$), then these densities grow to a singularity.

The total superenergies of the considered flat Friedmann universes are infinite for $t \in <0; \infty)$.

In the case $k = (-)1, p = \epsilon/3$ we have the same situation as in the case $k = 0$, but the densities ${}^{g}\epsilon_s$, ${}^{m}\epsilon_s$, ϵ_s are more complicated in form than (3.5)–(3.10), and they will not be cited here.

4. CONCLUDING REMARKS

In this paper we have considered the canonical superenergy tensors in the framework of GR. We emphasized that although the gravitational field in GR *has no energy-momentum tensor*, one can introduce as a *substitute* for the tensor the canonical superenergy ${}^{g}S_i^{\ k}$. This tensor is obtained by special averaging of the gravitational canonical energy-momentum pseudotensor ${}_{E}t_i^{\ k}$.

We also introduced into GR the canonical *total superenergy tensor* $S_i^{\ k} = {}^{g}S_i^{\ k} + {}^{m}S_i^{\ k}$, matter and gravitation, obtained by the same averaging of the sum $T_i^{\ k} + {}_{E}t_i^{\ k}$.

We pointed out some possible applications of the canonical superenergy in GR, and applied the canonical superenergy tensors to the analysis of Friedmann universes and showed that the superenergy densities and the total superenergetic quantities are positive-definite for all the considered Friedmann models and that they *produce singularities* if the cosmic time $t \to 0^+$.

On the other hand, *energetic quantities*, such as energy and linear and angular momentum, calculated by using the covariant *Komar single-index complex* and (also covariant) Pirani expression for energy *are equal to zero* (locally and globally) for all the Friedmann universes (Garecki, 1995) and *cannot produce* any singularities unless the early universe was anisotropic and nonhomogeneous.

We conclude from this that there *may be no link* between Komar's quantities and the Pirani energy calculated for Friedmann universes and the Hawking–Penrose singularity theorem. However, there must be a link between the superenergetic quantities for Friedmann models and this theorem. The physical meaning of these important facts will be investigated in the future.

REFERENCES

Bel, L. (1958). *Comptes Rendus d l'Academie des Sciences*, **247**, 1094.

Bel, L. (1959). *Comptes Rendus d l'Academie des Sciences*, **249**, 1297.

Bel, L. (1962). *Cahiers de Physique*, **16**, 59.

Cattaneo, C. (1965). In *Proceedings of the International Conference on Relativistic Theories of Gravitation*, Vol. II, London.

Dańczura, D. (1984). Diploma Thesis, University of Szczecin, Szczecin [in Polish].

Faddeev, L. D. (1982). *Uspekhi Fiziki Nauk*, **136**, 435.

Garecki, J. (1973). Ph.D. Thesis, Jagellonian University, Cracow [in Polish].

Garecki, J. (1978). *Acta Physica Polonica B*, **9**, 291.

Garecki, J. (1979). *Acta Physica Polonica B*, **10**, 883.

Garecki, J. (1980). *Acta Physica Polonica B*, **11**, 225.

Garecki, J. (1981a). In *Proceedings: Einstein Centenary Symposium*, Vol. II, Duhita Publishers c/o Einstein Foundation International, Nagpur.

Garecki, J. (1981b). *Acta Physica Polonica B*, **12**, 739.

Garecki, J. (1993). *Reports on Mathematical Physics* (Toruń), **33**, 57.

Garecki, J. (1995). *General Relativity and Gravitation*, **27**, 55.

Goldberg, J. N. (1980). In *General Relativity and Gravitation*, Vol. I, A. Held, ed., Plenum Press, New York.

Komar, A. (1959). *Physical Review*, **113**, 934.

Komar, A. (1962). *Physical Review*, **127**, 1411.

Kościelak, E. (1987). Diploma Thesis, University of Szczecin, Szczecin [in Polish].

Krawczak, H. (1982). Diploma Thesis, University of Szczecin, Szczecin [in Polish].

Landau, L. D., and Lifschitz, E. M. (1988). *Classical Field Theory*, Science Publishers, Moscow [in Russian].

Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation*, Freeman, San Francisco.

Møller, C. (1966). Survey of investigations on the energy-momentum complex in general relativity, Nordita publications No. 190, Copenhagen.

Møller, C. (1978). On the crisis in the theory of gravitation and a possible solution, *Danske Videnskabernes Selskab, Matematisk-Fysiske Meddeleleser*, **39**, 13.

Öktem, F. (1968). *Nuovo Cimento*, **58B**, 167.

Petrov, A. Z. (1966). *New Methods in General Relativity*, Moscow [in Russian].

Pirani, F. A. E. (1957). *Physical Review*, **105**, 1089.

Schouten, J. A. (1954). *Ricci-Calculus*, Springer-Verlag, Berlin.

Thirring, W. (1978). *Lehrbuch der Matematischen Physik, Vol. 2: Klassische Feldtheorie*, Springer-Verlag, Berlin.

Trautman, A. (1962). In *Gravitation: An Introduction to Current Research*, L. Witten, ed., New York.

Veblen, O. (1933). *Invariants of Quadratic Differential Forms*, Cambridge University Press, Cambridge.

Wald, R. (1984). *General Relativity*, University of Chicago Press, Chicago.

Weber, J. (1961). *General Relativity and Gravitational Waves*, Wiley-Interscience, New York.

Winicour, J. (1980). In *General Relativity and Gravitation*, Vol. I, A. Held, ed., Plenum Press, New York.